Honors Algorithms
G22.3520-001 Fall 2007

Lecture 13

Union/Find with “up trees”

Each set is implemented as a tree

Every node in the tree, other than the root, has a pointer “up” to its parent

The representative of a set is the root of its tree

Find: follow pointers to the root

Union: Merge one root into the other root

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Example of Union:

Assume n items and m operations

Worst case: mn — trees may degenerate into lists

Two simple ideas: size balancing and path compression

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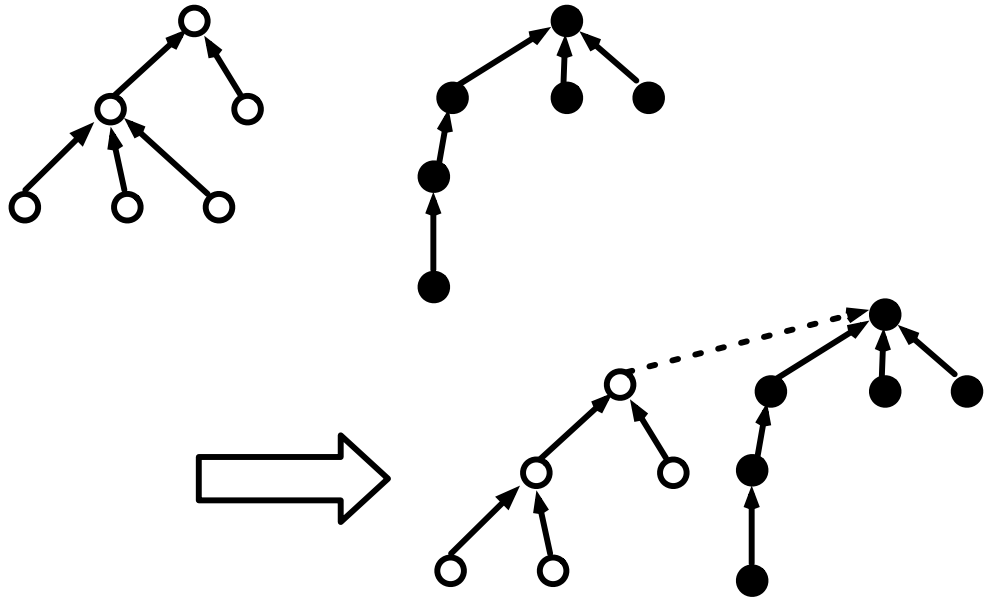

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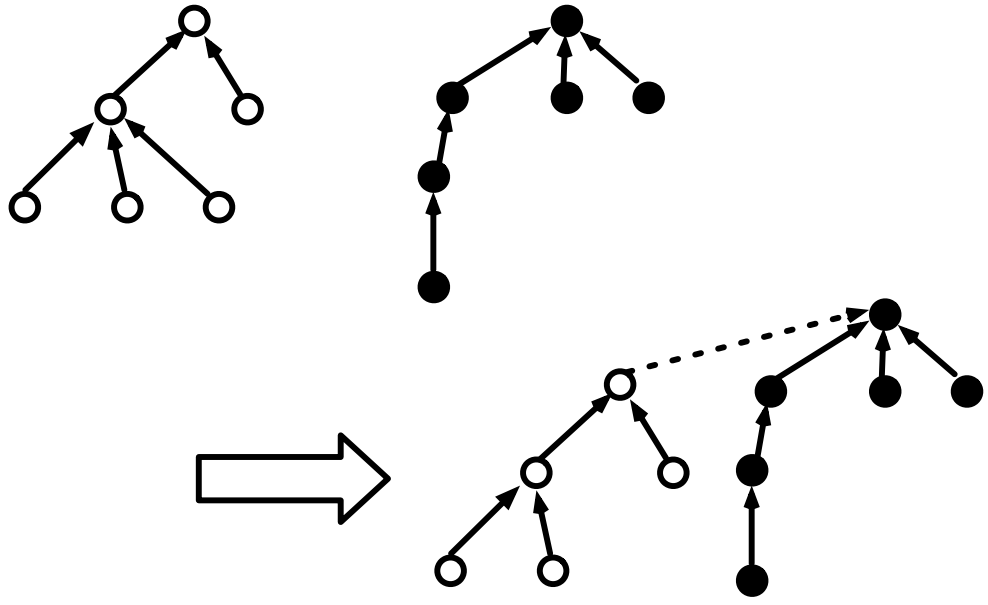

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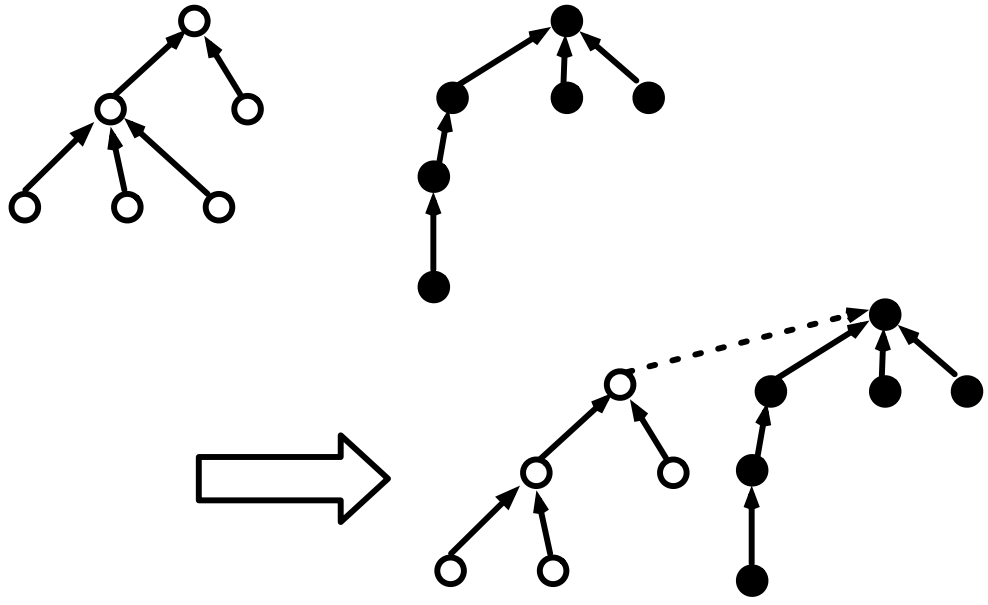

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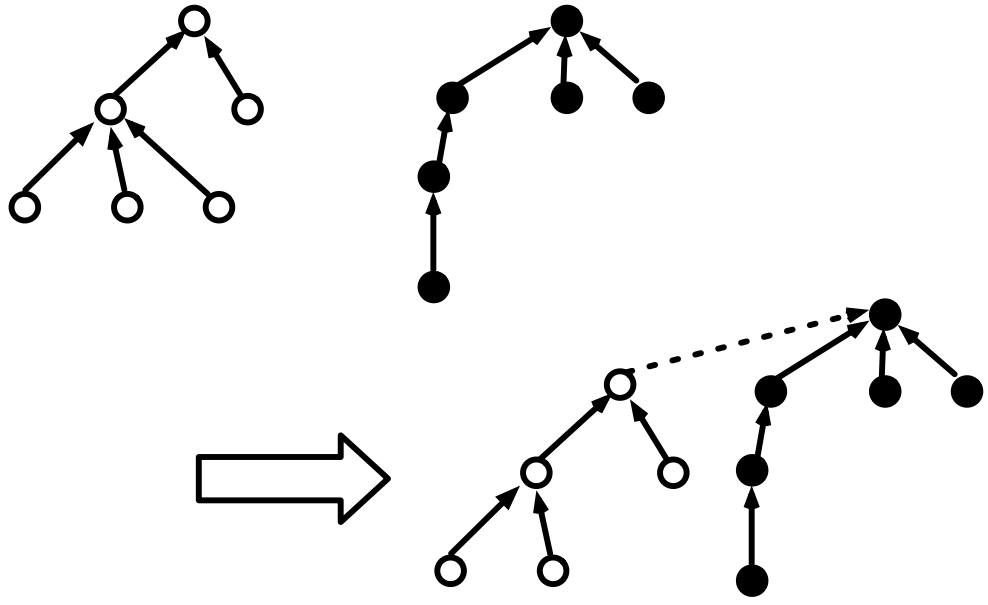

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Size balancing

Size balancing rule

In a Union operation, always merge the smaller tree into the larger tree

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Proof: induction on n.

Assume T was obtained by merging T_1 into T_2

where $n_1 := Size(T_1) \leq n_2 := Size(T_2)$

Let $h_i := Height(T_i)$ for $i = 1, 2$

By induction, $n_1 \geq 2^{h_1}$ and $n_2 \geq 2^{h_2}$

If $h_1 \geq h_2$, then $h = h_1 + 1$ and

$$n = n_1 + n_2 \geq 2n_1 \geq 2 \cdot 2^{h_1} = 2^h$$

If $h_1 < h_2$, then $h = h_2$ and

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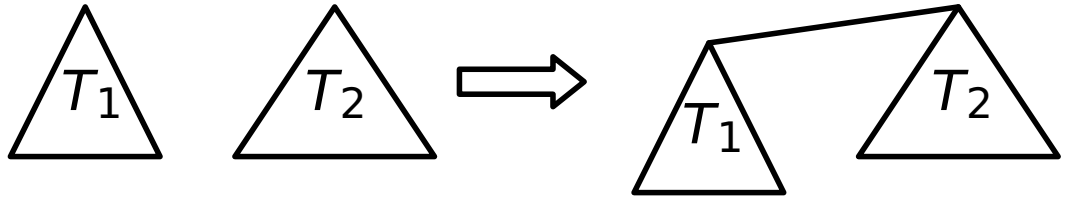

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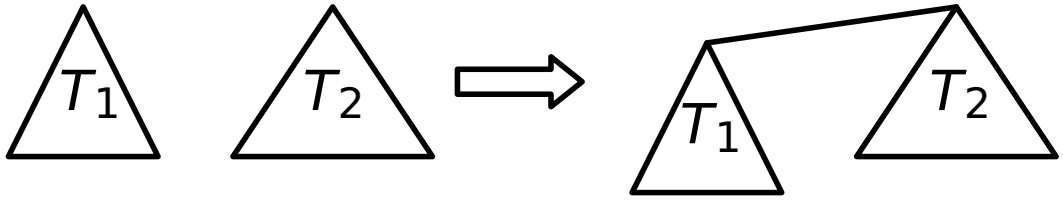

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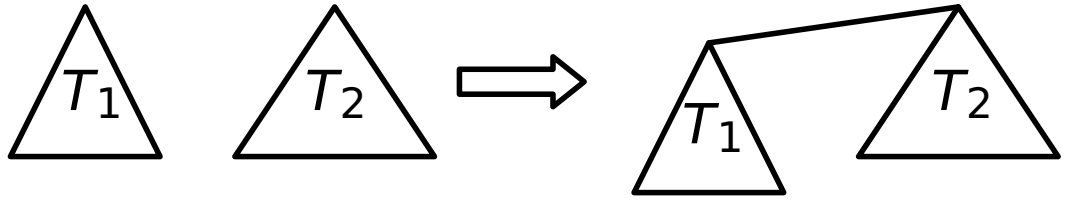

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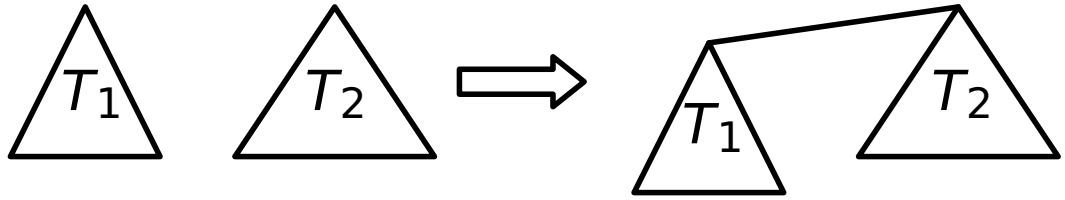

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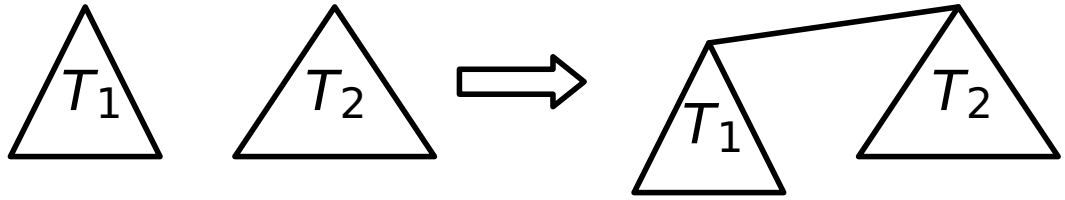

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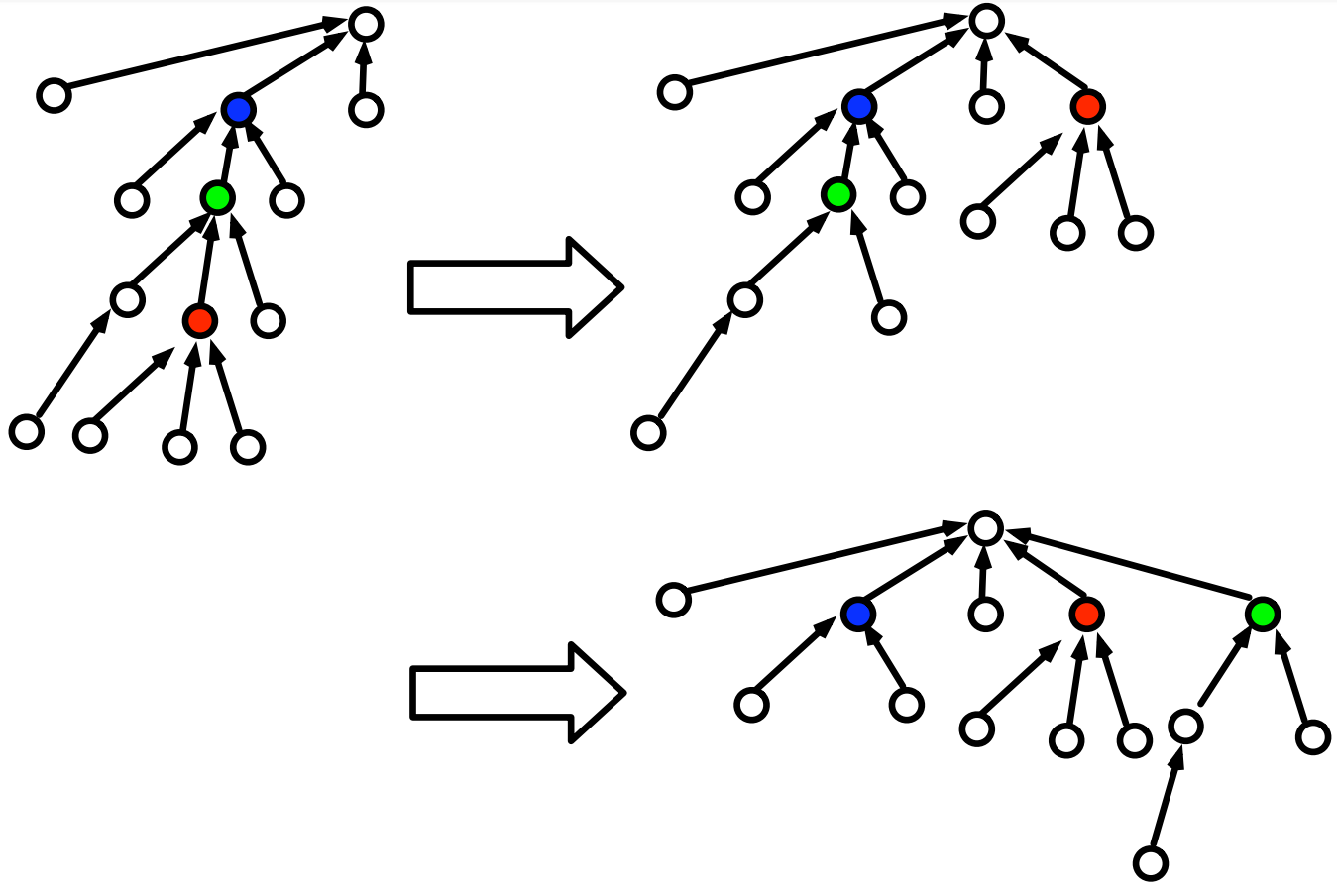

Running time analysis

For $g \geq 0$, define

$$F(g) := 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}} \Bigg\} \; g \text{ 2's}$$

Formally, $F(0) := 1$ and $F(g+1) := 2^{F(g)}$

Define $\log^* r :=$ least g such that $F(g) \geq r$

Theorem

With size balancing and path compression, any sequence of m union/find operations on n items takes time $O((m+n)\log^* n)$

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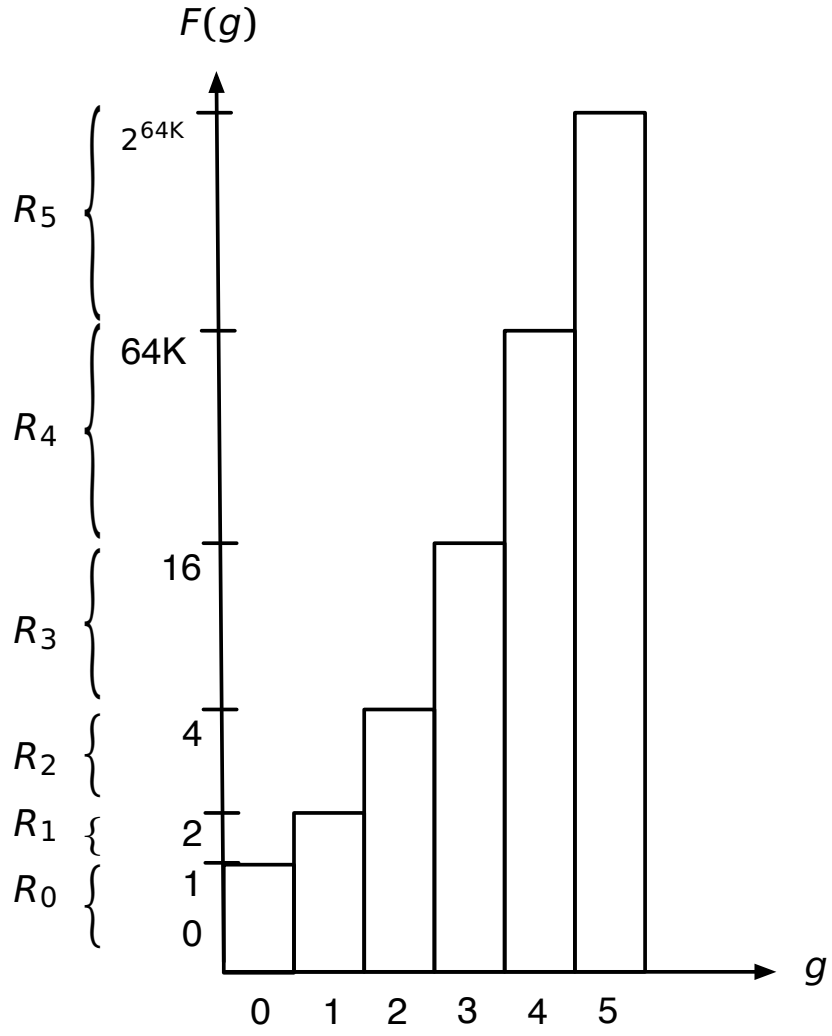
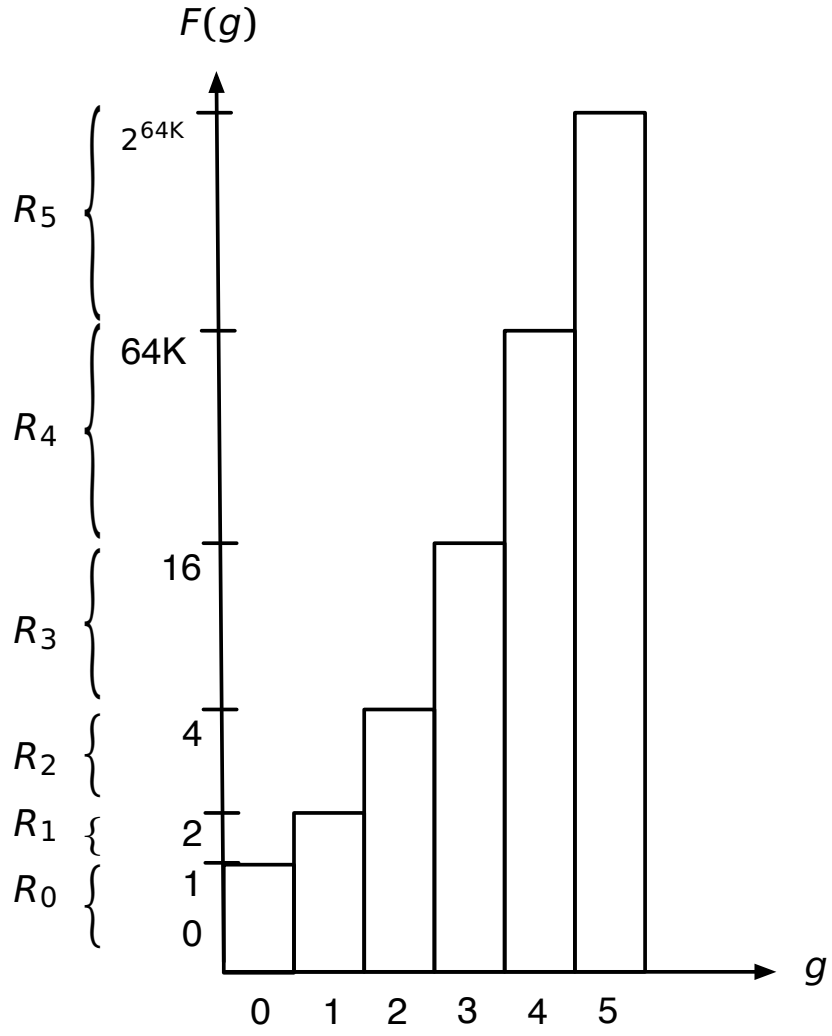


$$R_g := (\log^*)^{-1}(g) = \{r : \log^* r = g\}$$

$$\log^* r \leq g \Leftrightarrow r \leq F(g)$$

$$R_0 = \{0, 1\}, R_g = \{F(g-1)+1, \ldots, F(g)\} \text{ for } g > 0$$



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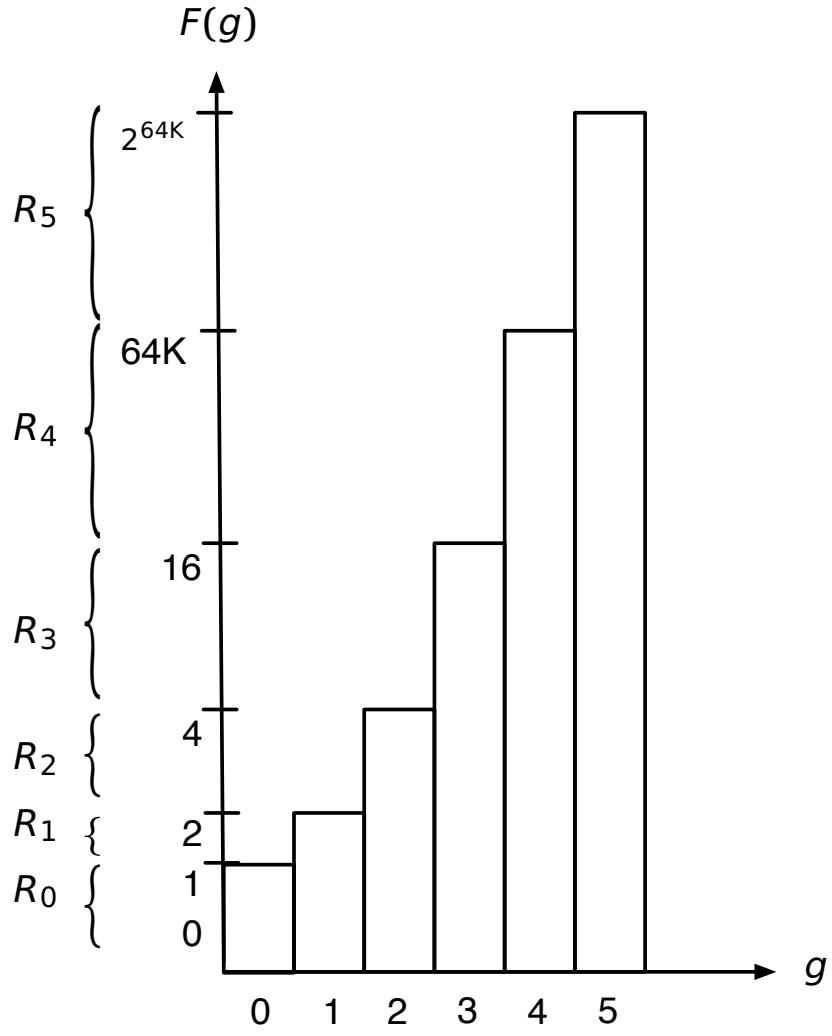

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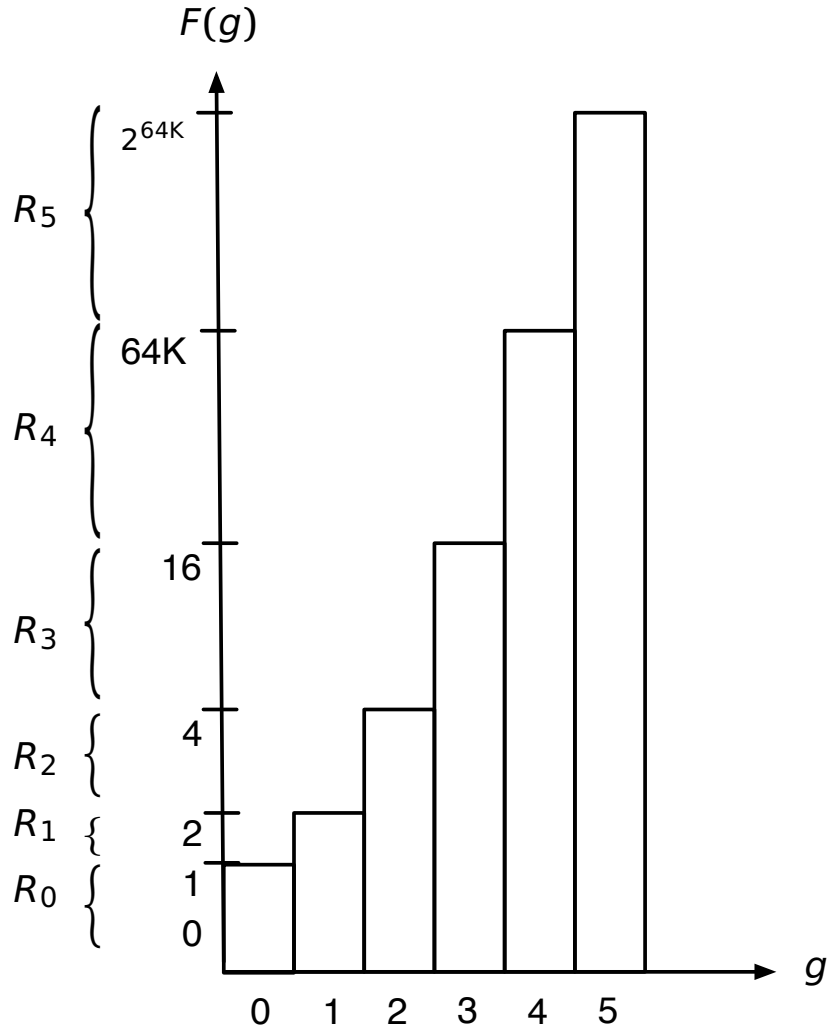

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Let $Op_1, \ldots, Op_m$ be a sequence of union/find operations

Consider the forest of trees $\mathcal{F}$ that results after executing $Op_1, \ldots, Op_m$ with size balancing, but no path compression

Define the rank of a node v to be its height in $\mathcal{F}$

rank is a static quantity – it does not change over time

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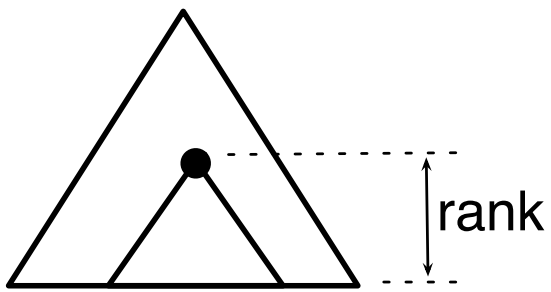

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Lemma 2

For every $r \geq 0$, there are at most $n/2^r$ nodes of rank r

Proof:

- By Lemma 1, any node of rank r is the root of a subtree in $\mathcal{F}$ of size $\geq 2^r$
- Any two distinct nodes of rank r are roots of disjoint subtrees in $\mathcal{F}$
- Therefore, there can be at most $n/2^r$ nodes of rank r
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Lemma 3

Suppose that at some time during the execution of $Op_1, \ldots, Op_m$ with compression, v is a (strict) descendent of w. Then $Rank(v) < Rank(w)$

Proof:

- Key observations:
- path compression only eliminates descendency relations – it never creates any new ones
- with no path compression, union/find operations never destroy descendency relations

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Prove by induction on i:

- for all v, w, if v is a descendent of w after executing $Op_1, \ldots, Op_i$ with compression, then v is a descendent of w after executing $Op_1, \ldots, Op_i$ without compression

Thus, if v is a descendent of w at some point in time during the execution of $Op_1, \ldots, Op_m$ with compression, then v is a descendent of w in $\mathcal{F}$, and hence $Rank(v) < Rank(w)$ – QED

Definition

For a node v, we define its group as $G(v) := \log^* Rank(v)$

Clearly, $G(v) \leq \log^* n$

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Proof of Theorem

Union operations take $O(1)$, so we can focus on find operations

Let $\mathcal{I}$ be the set of indices i such that Op_i is a find operation

Consider a fixed $i \in \mathcal{I}$, with $Op_i = \text{“}Find(v)\text{”}$

Consider the path from v to the root:

$$v = \underbrace{v_1, v_2, \ldots, v_{k-2}}_{\text{moved nodes}}, v_{k-1}, v_k = \text{root}$$

By Lemma 3, we have

$$Rank(v_1) < Rank(v_2) < \cdots < Rank(v_k)$$

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Proof of Theorem

Union operations take $O(1)$, so we can focus on find operations

Let $\mathcal{I}$ be the set of indices i such that Op_i is a find operation

Consider a fixed $i \in \mathcal{I}$, with $Op_i =$ "$Find(v)$"

Consider the path from v to the root:

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$C := \sum_{i \in \mathcal{I}} |X_i|$ is the cost of all the find operations

Let's split X_i into 3 sets:

- $Y_i := \{v_j : j < k-1 \text{ and } G(v_j) = G(v_{j+1})\}$
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So we have

$$C = \sum_{i \in \mathcal{I}} (|Y_i| + |Z_i| + |W_i|)$$

$$\leq \sum_{i \in \mathcal{I}} |Y_i| + m \log^* n + 2m$$

Claim: $C' := \sum_i |Y_i| \leq n \log^* n$

Idea: Consider a fixed node v

- Each time v moves during a path compression, v's new parent has a higher rank than v's old parent
- If $G(v) = g$, then after $|R_g| - 1$ moves, v must acquire a parent whose group is $> g$

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For $g \geq 0$, let $V_g := \{v : G(v) = g\}$

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$$C' \leq \sum_{g=0}^{\log^* n} |V_g| \cdot (|R_g| - 1)$$

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To prove the claim, it will suffice to show that

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